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Common Solutions of the Einstein and Brans-Dicke Theories

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Abstract

It is shown that non-trivial solutions common to the vacuum field equations of the Einstein and of the Brans-Dicke theories necessarily represent *pp*-waves and the set of all common solutions is precisely the set of all *pp*-wave solutions of the Einstein equations. The form of the associated scalar field is found and is shown to be singular when $\omega < -1$.

Horndeski (1973) has shown that the plane-wave solutions of Einstein's vacuum equations found by Kundt (1961) are also solutions of the vacuum field equations of the Brans-Dicke theory (Brans & Dicke, 1962). Kundt's solutions are of type N in the Petrov classification and represent the so-called *pp*-waves (Ehlers & Kundt, 1962). In this note we show that, apart from trivial flat space-time solutions, all common vacuum solutions of the Einstein and Brans-Dicke theories represent *pp*-waves and these common solutions consist of all possible *pp*-wave solutions of Einstein's equations. We ignore the case ϕ = constant for which the Brans-Dicke equations reduce to those of Einstein.

The vacuum field equations of the Brans-Dicke theory are

$$R_{\mu\nu} + \phi^{-1}(\phi_{;\mu\nu} + \tfrac{1}{2}g_{\mu\nu}\phi_{;\ \alpha}^{\ \alpha}) + \omega\phi^{-2}\phi_{,\mu}\phi_{,\nu} = 0 \tag{1}$$

$$(3 + 2\omega)\phi_{;\ \alpha}^{\ \alpha} = 0 \tag{2}$$

and the required common solutions will also satisfy the Einstein vacuum equations

$$R_{\mu\nu} = 0 \tag{3}$$

From equations (1), (2) and (3) it follows that when $\omega \neq 0$ and $\omega \neq -\frac{3}{2}$ the common solutions are derived from the equations

$$\phi_{;\mu\nu} + \omega\phi^{-1}\phi_{,\mu}\phi_{,\nu} = 0 \tag{4}$$

$$\phi_{,\mu}\phi_{,}^{\ \mu} = 0 \tag{5}$$

together with equation (3). When $\omega = 0$ or $\omega = -\frac{3}{2}$, $\phi_{,\mu}$ may be a null vector, as in equation (5), but is not necessarily so. Assuming that $\phi_{,\mu}$ is a null vector the equations (4) and (5) hold for all ω. A vector τ_μ can be defined by

$$\tau_\mu = \phi^\omega \phi_{,\mu} \tag{6}$$

so that equations (4) and (5) become

$$\tau_{\mu;\nu} = 0 \tag{7}$$

$$\tau_\mu \tau^\mu = 0 \tag{8}$$

Thus the space-time admits a covariantly constant null vector and, by a theorem of Ehlers & Kundt (1962), this characterizes a *pp*-wave solution.

That all *pp*-wave solutions of the Einstein equations are included in these common solutions can be seen from the fact that any *pp*-wave solution of equations (3) admits a vector τ_μ satisfying equations (7) and (8) and since this vector is a gradient (Ehlers & Kundt, 1962) it follows that equation (6) can be integrated to give a function ϕ satisfying equations (4) and (5).

There remains the case when $\omega = 0$ or $\omega = -\frac{3}{2}$ and $\phi_{,\mu}$ is not a null vector. Consider first $\omega = 0$. From equations (1), (2) and (3) we have $\phi_{;\mu\nu} = 0$, i.e.

$$\phi_{;\mu\nu\sigma} - \phi_{;\mu\sigma\nu} \equiv R^\alpha{}_{\mu\nu\sigma}\phi_{,\alpha} = 0 \tag{9}$$

In view of equation (3) this can be written in the form

$$C_{\alpha\mu\nu\sigma}\phi_{,}^{\ \alpha} = 0 \tag{10}$$

where $C_{\alpha\mu\nu\sigma}$ is the Weyl tensor. Taking the right dual of equation (10) and using the fact that the right and left duals of the Weyl tensor are equal we obtain

$$\eta_{\alpha\beta\gamma\delta}C^{\gamma\delta}{}_{\mu\nu}\phi_{,}^{\ \alpha} = 0$$

where $\eta_{\alpha\beta\gamma\delta}$ is the permutation tensor. Multiplying this by $\eta^{\beta\epsilon\rho\sigma}$ we find

$$\delta^{\epsilon\rho\sigma}_{\alpha\gamma\delta}C^{\gamma\delta}{}_{\mu\nu}\phi_{,}^{\ \alpha} = 0$$

i.e.

$$\phi_{,}^{\ [\epsilon}C^{\rho\sigma]}{}_{\mu\nu} = 0$$

Contracting this expression with $\phi_{,\epsilon}$ and using equation (10) we obtain finally

$$\phi_{,\epsilon}\phi_{,}^{\ \epsilon}C^{\rho\sigma}{}_{\mu\nu} = 0$$

so that if $\phi_{,\epsilon}$ is not null, $C^{\rho\sigma}{}_{\mu\nu} = 0$ which, from (3), implies that the space-time is flat.

Consider now $\omega = -\frac{3}{2}$ with $\phi_{,\mu}$ not null. From equations (1), (2) and (3) we have

$$\phi_{;\mu\nu} = \phi^{-1}(\tfrac{3}{2}\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{4}g_{\mu\nu}\phi_{,\alpha}\phi_{,}^{\alpha}) \tag{11}$$

Differentiating equation (10) we find, as in the case $\omega = 0$, that equation (8) holds so, by the same argument, the space-time is again flat. This concludes the proof of the stated result.

To investigate the form of the function ϕ corresponding to these common solutions we note that the conditions (7) and (8) imply the existence of a co-ordinate system in which the metric assumes the form (Ehlers & Kundt, 1962)

$$ds^2 = 2H(x, y, u)\, du^2 + 2du\, dv - dx^2 - dy^2 \tag{12}$$

with

$$H_{,xx} + H_{,yy} = 0$$

and in which the vector field τ_μ assumes the form

$$\tau_\mu = u_{,\mu} \tag{13}$$

Equation (13) implies that $\phi = \phi(u)$ and τ_μ is absolutely constant, i.e.

$$\phi^\omega \phi_{,u} = 1$$

which integrates to give

$$\phi = \phi_0 u^{(\omega+1)^{-1}} \qquad (\omega \neq -1) \tag{14}$$

$$\phi = \phi_0 e^u \qquad (\omega = -1) \tag{15}$$

with a suitable choice of origin, ϕ_0 being a constant.

From (14) ϕ is singular at $u = 0$ when $\omega < -1$; the physical meaning of this is obscure but the existence of singularities in the scalar ϕ associated with a singularity-free metric solution has been noticed elsewhere (O'Hanlon & Tupper, 1972). Another curious feature is the presence of flat space-time solutions associated with non-constant values of ϕ, which appears to be inconsistent with the statement (Brans & Dicke, 1962) 'ϕ has as its sources the matter distribution in space'. In particular, the solution (12) is flat when $H = 0$ and a corresponding solution for ϕ is that given by (14) or (15). Hence when $\omega < -1$, flat space-time with a singular scalar field is a solution of the Brans-Dicke vacuum field equations.

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References

Brans, C. and Dicke, R. H. (1961). *Physical Review,* **124,** 925.
Ehlers, J. and Kundt, W. (1962). *Gravitation: An Introduction to Current Research* (Ed. L. Witten). Wiley, New York.
Horndeski, G. W. (1973). Ph.D. Thesis, University of Waterloo.
Kundt, W. (1961). *Zeitshrift für Physik,* **163,** 77.
Newman, E. T. and Penrose, R. (1962). *Journal of Mathematical Physics,* **3,** 566.
O'Hanlon, J. and Tupper, B. O. J. (1972). *Il Nuovo Cimento,* **7B**, 305.